

# Generalized Gibbons-Hawking-York term for $f(R)$ gravity

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## Abstract

A generalization to the Gibbons-Hawking-York boundary term for metric  $f(R)$  gravity theories is introduced. A redefinition of the Gibbons-Hawking-York term is proposed. The proposed new definition is used to derive a consistent set of field equations and is extended to metric  $f(R)$  gravity theories. The surface terms in the action are gathered into a total variation of some quantity. A total divergence term is added to the action to cancel these terms. Finally, the new definition is proven to demand no restrictions on the value of  $\delta g_{ab}$  or  $\partial_c \delta g_{ab}$  on the boundary.

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## 1. Introduction

There are three popular formulations for  $f(R)$  gravity theories: metric  $f(R)$  gravity, metric-affine  $f(R)$  gravity and Palatini  $f(R)$  gravity. Metric  $f(R)$  has been the most popular of the three, mainly because it passes all the observational and theoretical constraints [1]. Recently, metric  $f(R)$  gravity has proven quite useful in providing a toy model for studying the problem of dark energy. One problem with metric  $f(R)$  that has received less attention is the problem of fixing the surface terms on the boundary when deriving the field equations. One possible reason for this is that unlike in General Relativity, the boundary terms do not consist of a total variation of some quantity. Therefore, it is not possible to add a total divergence to the action in order to heal the problem and arrive at a well meaning variational principle [2]. In this paper, we will show that in fact the boundary terms can be expressed as a total variation of some quantity. We will present a redefinition of the Gibbons-Hawking-York [5, 6] boundary term to cancel these surface terms and arrive at a consistent set of field equations. It has been widely accepted that the modification of the gravitational action with respect to the surface term by adding the Gibbons-Hawking-York term to the total action is desirable because it cancels the terms involving  $\partial \delta g_{ab}$ , and so setting  $\delta g_{ab} = 0$  becomes sufficient to make the action stationary. In order to see the role Gibbons-Hawking-York boundary term we

must first review its role in general relativity. If we consider the Einstein-Hilbert action [3]

$$S_{EH} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} R, \quad (1)$$

where  $R = R_{ab}g^{ab}$  is the Ricci scalar.

The variation of (1) with respect to  $\delta g_{ab}$  gives

$$\delta S_{EH} = \frac{1}{2\kappa} \int d^4x (\delta \sqrt{-g} R + \sqrt{-g} \delta R), \quad (2)$$

using  $\delta \sqrt{-g} = -\frac{1}{2} \sqrt{-g} g_{ab} \delta g^{ab}$  and  $\delta R_{ab} = \nabla_c \delta \Gamma_{ab}^c - \nabla_b \delta \Gamma_{ac}^c$ , the variation  $\delta S_{EH}$  becomes

$$\begin{aligned} \delta S_{EH} = & \int_{\nu} d^4x \left( -\frac{1}{2} R g_{ab} \sqrt{-g} \delta g^{ab} + R_{ab} \sqrt{-g} \delta g^{ab} \right. \\ & \left. + \sqrt{-g} g^{ab} (\nabla_c \delta \Gamma_{ab}^c - \nabla_b \delta \Gamma_{ac}^c) \right). \end{aligned} \quad (3)$$

Finally, using the fact that the covariant derivative of the metric tensor vanishes in general relativity, we can rewrite the third term as the divergence of a vector field  $J^c$ , where  $J^c = g^{ab} \delta \Gamma_{ab}^c - g^{ac} \delta \Gamma_{ab}^b$ . Using Gauss-Stokes theorem [7], the variation (3) becomes

$$\begin{aligned} \delta S_{EH} = & \int_{\nu} d^4x \sqrt{-g} (R_{ab} - \frac{1}{2} R g_{ab}) \delta g^{ab} \\ & + \oint_{\partial \nu} d^3x \sqrt{|h|} n_c J^c, \end{aligned} \quad (4)$$

where  $h$  is the induced metric tensor associated with hypersurface and  $n_c$  is the normal unit vector on hypersurface  $\partial \nu$ . The first term in (4) is the familiar Einstein field equations multiplied by  $\delta g^{ab}$ , however in order to derive the field equations, the action must be stationary and consequently the second term must vanish. We can cancel the second term by adding a boundary term to the total action in (1) that cancels this surface term.

Using the definition of the christoffel symbol (i.e.  $\Gamma_{bc}^a = \frac{1}{2} g^{ad} (\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc})$ ),  $J_c$  becomes

$$J_c = g^{ab} (\nabla_b \delta g_{ca} - \nabla_c \delta g_{ba}), \quad (5)$$

substituting the completeness relation  $g^{ab} = h^{ab} - n^a n^b$ , so that

$$\begin{aligned} n^c J_c &= n^c (h^{ab} - n^a n^b) (\nabla_b \delta g_{ca} - \nabla_c \delta g_{ba}) \\ &= n^c h^{ab} \nabla_b \delta g_{ca} - n^c h^{ab} \nabla_c \delta g_{ba} \\ &\quad - n^a n^b n^c \nabla_b \delta g_{ca} + n^a n^b n^c \nabla_c \delta g_{ba}. \end{aligned} \quad (6)$$

The third and fourth terms vanish because of the antisymmetry of  $(\nabla_b \delta g_{ca} - \nabla_c \delta g_{ba})$ , Consequently we obtain

$$n^c J_c|_{\partial\nu} = n^c h^{ab} (\partial_b \delta g_{ca} - \partial_c \delta g_{ba}). \quad (7)$$

Since  $\delta g_{ab}$  vanishes everywhere on  $\partial\nu$ , its tangential derivative must also vanishes. It follows that  $h^{ab} \partial_b \delta g_{ca} = 0$  and we obtain

$$n^c J_c|_{\partial\nu} = -n^c h^{ab} \partial_c \delta g_{ba}, \quad (8)$$

substituting (8) in (4), the last term becomes

$$- \oint_{\partial\nu} d^3x \sqrt{|h|} n^c h^{ab} \partial_c \delta g_{ba}. \quad (9)$$

Now, the trick is to construct a term that cancels this term. Gibbons, Hawking and York proposed adding the trace of the extrinsic curvature of the boundary to the action. If we consider the Gibbons-Hawking-York boundary term

$$S_{GHY} = 2 \oint_{\partial\nu} d^3x \sqrt{|h|} K, \quad (10)$$

where  $K = \nabla_c n^c$  is intrinsic curvature, using  $\delta K|_{\partial\nu} = n^c h^{ab} \partial_c \delta g_{ba}$ , the variation of this term gives

$$\delta S_{GHY} = \oint_{\partial\nu} d^3x \sqrt{|h|} n^c h^{ab} \partial_c \delta g_{ba}, \quad (11)$$

Notice that this boundary term cancels (9) totally. The role of the Gibbons-Hawking-York is crucial when considering the Einstein-Hilbert action since it is the basis for the most elementary variational principle from which the field equations of general relativity can be defined. However, the use of the Einstein-Hilbert action is appropriate only when the underlying spacetime manifold  $\nu$  is closed, i.e. a manifold which is both compact and without boundary. In the event that the manifold has a boundary  $\partial\nu$ , the action should be supplemented by a boundary term so that the variational principle is well-defined.

## 2. Redefining the function of Gibbons-Hawking-York term

Generally, the Gibbons-Hawking-York term is defined as the term added to the total action to cancel the surface terms if the action is to be stationary. However, if we redefine the Gibbons-Hawking-York term as the term add to the Ricci scalar, such that

$$\begin{aligned} & \int d^4x \sqrt{-g} R + 2 \oint_{\partial\nu} d^3x \sqrt{|h|} K \\ &= \int d^4x \sqrt{-g} \tilde{R}, \end{aligned} \quad (12)$$

where  $\tilde{R} = R - \phi$  and  $\phi = 2\nabla_c(h^{ab}\Gamma_{ab}^c)$ . The variation of (12) gives

$$\begin{aligned}\delta \int d^4x \sqrt{-g} \tilde{R} &= \int d^4x \sqrt{-g} (\delta R - \delta \phi - \frac{1}{2} g_{ab} \delta g^{ab} \tilde{R}) \\ &= \int d^4x \sqrt{-g} (R_{ab} - \frac{1}{2} g_{ab} R) \delta g^{ab} \\ &\quad + \oint d^3x \sqrt{|h|} (J^c n_c - 2h^{ab} \delta \Gamma_{ab}^c n_c),\end{aligned}\tag{13}$$

where we have used

$$\begin{aligned}&\int d^4x \sqrt{-g} \delta \phi - \int d^4x \sqrt{-g} \frac{1}{2} g_{ab} \delta g_{ab} \phi \\ &= \delta \int d^4x (2\sqrt{-g} \nabla_c (h^{ab} \Gamma_{ab}^c)) = 2 \oint d^3x \sqrt{|h|} h^{ab} \delta \Gamma_{ab}^c n_c.\end{aligned}\tag{14}$$

Using the fact that the quantity  $J^c n_c - h^{ab} \delta \Gamma_{ab}^c n_c$  vanishes at the boundary, the variation of (12) gives the familiar field equations without boundary terms, i.e.

$$\delta \int d^4x \sqrt{-g} \tilde{R} = \int d^4x \sqrt{-g} (R_{ab} - \frac{1}{2} R g_{ab}) \delta g^{ab},\tag{15}$$

It's clear that the modification considered in (12) does the same job as the Gibbons-Hawking-York term without the need to consider the vanishing of  $\delta g_{ab}$  at the boundary of the hypersurface. Although this may seem simple, in the next section we show that this new definition is crucial in deriving the field equations for  $f(R)$  metric gravity without the need to worry about any surface terms that need be canceled.

### 3. Variational principle in metric $f(R)$ gravity

If we consider the modified  $f(R)$  gravity action

$$S_R = \int d^4x \sqrt{-g} f(R),\tag{16}$$

the variation of this action

$$\delta S_R = \int d^4x (\delta \sqrt{-g} f(R) + \sqrt{-g} f'(R) \delta R),\tag{17}$$

using  $\delta R = \delta g^{ab} R_{ab} + g_{ab} \nabla_c \nabla^c (\delta g^{ab}) - \nabla_a \nabla_b (\delta g^{ab})$  and  $\delta \sqrt{-g} = -\frac{1}{2} \sqrt{-g} g_{ab} \delta g^{ab}$ , (17) becomes

$$\begin{aligned}\delta S_R &= \int d^4x \sqrt{-g} (f'(R) \delta g^{ab} R_{ab} + f'(R) g_{ab} \nabla_c \nabla^c (\delta g^{ab}) \\ &\quad - f'(R) \nabla_a \nabla_b (\delta g^{ab}) - \frac{1}{2} f(R) g_{ab} \delta g^{ab}).\end{aligned}\tag{18}$$

To derive the field equations we must reexpress the second and third terms as  $\delta g^{ab}$  multiplied by some quantity. To do that we define

$$\begin{aligned} W^c &= f'(R)[g_{ab}g^{cd}\nabla_d\delta g^{ab} - \nabla_a\delta g^{ca}] \\ &\quad - \delta g^{ab}[g_{ab}g^{cd}\nabla_d f'(R) - \delta_b^c\nabla_a f'(R)]. \end{aligned} \quad (19)$$

Rewriting the second and third terms in (18) in terms of  $\nabla_c W^c$  and using Gauss-Stokes theorem on the last term, (18) becomes

$$\begin{aligned} \delta S_R &= \int d^4x \sqrt{-g}(f'(R)R_{ab} + g_{ab}\nabla_c\nabla^c(f'(R)) \\ &\quad - \nabla_a\nabla_b(f'(R)) - \frac{1}{2}f(R)g_{ab})\delta g^{ab} \\ &\quad + \oint_{\partial\nu} d^4x \sqrt{|h|}n_c W^c. \end{aligned} \quad (20)$$

Evaluating the quantity  $W^c$  at the boundary yields

$$W^c|_{\partial\nu} = -f'(R)n^c h^{ab}\partial_c\delta g_{ba}. \quad (21)$$

If the action is to be stationary,  $W^c$  must be canceled with boundary term. One way to do that is to add the Gibbons-Hawking-York term multiplied by  $f'(R)$  to action in (16), such that

$$\begin{aligned} S_R + S_{GHYmod} &= \int d^4x \sqrt{-g}f(R) \\ &\quad + 2 \oint_{\partial\nu} d^3x \sqrt{|h|}f'(R)K. \end{aligned} \quad (22)$$

However, the variation of this term requires setting  $\delta R = 0$  on the boundary of the hypersurface [4], consequently this creates a strong condition on how  $\delta g^{ab}$  is varied near the hypersurface since  $\partial_c\delta g^{ab}$  is no longer arbitrary on the boundary. To overcome this restriction we use our new definition of the Gibbons-Hawking-York and replace  $f(R)$  by  $f(\tilde{R}) = f(R - \phi) = \sum_{n=0}^{\infty}(-\phi)^n \frac{d^n}{dR^n}f(R)$ , where  $\phi$  is now defined such that  $\delta\phi = \nabla_c\gamma_{ab}^c\delta g^{ab} - \delta g^{ab}\nabla_c\gamma_{ab}^c$ , with the operator  $\gamma_{ab}^c = g_{ab}\nabla^c - \delta_a^c\nabla_b$ . The variation of (16) gives

$$\begin{aligned} \delta S_R &= \delta \int d^4x \sqrt{-g} \sum_{n=0}^{\infty} (-\phi)^n \frac{d^n}{dR^n} f(R) \\ &= \delta \int d^4x \sqrt{-g} (\delta R - \delta\phi) f'(\tilde{R}) \\ &= \delta \int d^4x \sqrt{-g} ((\delta g^{ab}R_{ab} + g^{ab}\delta R_{ab} - \nabla_c\gamma_{ab}^c\delta g^{ab} \\ &\quad + \delta g^{ab}\nabla_c\gamma_{ab}^c) f'(\tilde{R}) - \frac{1}{2}f(\tilde{R})g_{ab}\delta g^{ab}). \end{aligned} \quad (23)$$

Using the definition of  $\gamma_{ab}^c$ , this variation becomes

$$\begin{aligned}\delta S_R &= \int d^4x \sqrt{-g} (f'(\tilde{R}) R_{ab} + g_{ab} \nabla_c \nabla^c (f'(\tilde{R}))) \\ &\quad - \nabla_a \nabla_b (f'(\tilde{R})) - \frac{1}{2} f(\tilde{R}) g_{ab} \delta g^{ab},\end{aligned}\tag{24}$$

for an arbitrary variation  $\delta g^{ab}$ , we get the field equation

$$\begin{aligned}f'(\tilde{R}) R_{ab} + g_{ab} \nabla_c \nabla^c (f'(\tilde{R})) \\ - \nabla_a \nabla_b (f'(\tilde{R})) - \frac{1}{2} f(\tilde{R}) g_{ab} = 0\end{aligned}\tag{25}$$

#### 4. Conclusions

We have redefined the Gibbons-Hawking-York term as the term add to the Ricci scalar and have shown that this definition works just as well as the original Gibbons-Hawking-York in illuminating any possible boundary terms so that we can have a well defined stationary action. This derivation requires no restrictions on the value of  $\delta g_{ab}$  or  $\partial \delta g_{ab}$  on the boundary of the manifold. Although, unlike the familiar metric  $f(R)$  field equation [2], (25) is a differential equation for  $f(R - \phi)$ . Rewriting the variation (23) in terms of  $f(R)$  yields

$$\begin{aligned}\delta S_R &= \int d^4x \sqrt{-g} (\delta R f'(R) - \frac{1}{2} f(R) g_{ab} \delta g^{ab} \\ &\quad - \delta \phi f'(R) - \delta R \phi f''(R) + \dots),\end{aligned}\tag{26}$$

the variation of  $S_R + S_{GHYmod}$  gives

$$\begin{aligned}\delta S_R + \delta S_{GHYmod} &= \int d^4x \sqrt{-g} (\delta R f'(R) \\ &\quad - \frac{1}{2} f(R) g_{ab} \delta g^{ab}) \\ &\quad + \oint d^3x \sqrt{|h|} (\delta K f'(R) \\ &\quad + K f''(R) \delta R),\end{aligned}\tag{27}$$

If the action  $S_R + S_{GHYmod}$  is to be stationary, the last term must vanish (i.e. setting  $\delta R = 0$  on the boundary). However, in (26) there is no need to set such a condition on the boundary since the second and higher derivatives of  $f(R)$  give us a series of field equations for  $f(R)$  and its derivatives that can be gathered to give one field equation for  $f(\tilde{R})$ .

In conclusion, the Gibbons-Hawking-York term must be an infinite number of boundary integrals in terms of the first and higher derivatives of  $f(R)$  that are added to the total action to cancel the surface terms. The reason that there is

only one boundary integral for the Einstein-Hilbert action is because the second and higher derivatives of  $f(R)$  vanish.  $S_{GHY_{mod}}$  require setting  $\delta R = 0$  because we have not considered the rest of the boundary integrals.

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